

Exploring the Influence of Graph Operations on Zero Forcing Sets

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Abstract

Zero forcing in graphs is a coloring process where a colored vertex can *force* its unique uncolored neighbor to be colored. A zero forcing set is a set of initially colored vertices capable of eventually coloring all vertices of the graph. In this paper, we focus on the numbers $z(G; i)$, which is the number of zero forcing sets of size i of the graph G . These numbers were initially studied by Boyer et al. where they conjectured that for any graph G on n vertices, $z(G; i) \leq z(P_n; i)$ for all $i \geq 1$ where P_n is the path graph on n vertices. The main aim of this paper is to show that several classes of graphs, including outerplanar graphs and threshold graphs, satisfy this conjecture. We do this by studying various graph operations and examining how they affect the number of zero forcing sets.

Keywords: Zero forcing set, path graph, tree, outerplanar graph, threshold graph

MSC2020: 05C05, 05C15, 05C30, 05C69

1 Introduction

Suppose we are given a simple graph G where some vertices are colored. The *zero forcing rule* is the following: If there is a colored vertex v such that its neighborhood contains a unique uncolored vertex w , then w can be colored. A *zero forcing set* of G is a set of vertices S such that if we start with the vertices in S colored, we can eventually color all vertices of G by repeatedly applying the zero forcing rule. We sometimes simply say the set S is forcing in G . If a vertex v is used to color a vertex w , we say that v *forces* w .

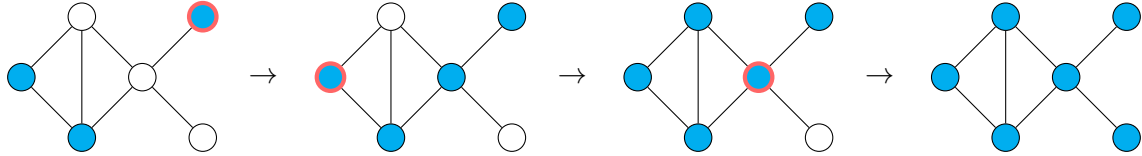


Figure 1: Zero forcing in a graph with the forcing vertex highlighted red at each step.

For a graph G and $i \geq 1$, we set $Z(G; i)$ to be the collection of forcing sets of G of cardinality i . Similarly, $Z'(G; i)$ is the set of non-forcing sets of G of cardinality i . We also set $z(G; i) = \#Z(G; i)$ and $z'(G; i) = \#Z'(G; i)$. Note that $z(G; i) + z'(G; i) = \binom{n}{i}$ where n

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is the number of vertices of G . The *zero forcing number* of G , denoted $z(G)$, is the minimum cardinality of a forcing set of G , i.e., $z(G) = \min\{i \mid z(G; i) \neq 0\}$.

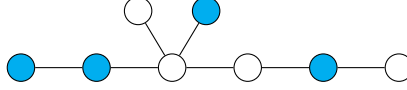


Figure 2: A non-forcing set in a graph.

The study of zero forcing sets in graphs has garnered significant interest due to its role as a coloring process on graphs and its intriguing connections to various fields such as linear algebra [2, 19, 22], power grid domination strategies [4, 6, 20], theoretical computer science [11, 29], modeling various physical phenomena [8, 12, 14], control of quantum systems [9], and rumor spreading models [16]. Initially introduced by Burgarth and Giovannetti [9], the zero forcing process has since been examined extensively by researchers. The concept of zero forcing number, introduced by an AIM research group, serves as a bound for the maximum nullity of a graph [19] and has connections with other graph parameters as well (see [3, 10, 24]). The problem of zero forcing has also been studied in randomized setups [13, 16, 23] and from a reconfiguration perspective [17]. A few other variants of zero forcing have also been well explored (see [1, 7, 15, 18, 21]).

The counting problem associated with zero forcing sets was first explored by Boyer et al. in [5]. Here, the authors studied the numbers $z(G; k)$ for various classes of graphs, such as complete graphs, cycle graphs, threshold graphs, and more. However, the case of path graphs is of particular interest due to the following conjecture, which asserts that path graphs *dominate* all other graphs in terms of number of forcing sets.

Conjecture 1.1. [5, Question 2] *For any graph G on n vertices, $z(G; i) \leq z(P_n, i)$ for all $i \geq 1$ where P_n is the path graph on n vertices.*

There are relatively few results about graphs that satisfy this conjecture. It is known that graphs that have a Hamiltonian path or a fort of appropriate size (see Definition 2.2) satisfy this conjecture [5, Propositions 17, 18]. It is also known that graphs with a certain bound on the minimum degree of the vertices satisfy the conjecture (see [13, Corollary 1.10]). One of the principle aims of this paper is to expand the class of graphs known to satisfy Conjecture 1.1.

We prove our results by studying various graph operations (such as deleting certain vertices or edges) and examining their effect on forcing sets. Using these operations, we obtain bounds on the numbers $z(G; i)$, which we then use to show that the graph G satisfies Conjecture 1.1.

In particular, we show that trees satisfy Conjecture 1.1, which was a problem suggested by Sam Spiro [26, Conjecture 3.7]. We in fact prove the following stronger result.

Theorem 1.2. *All outerplanar graphs (see Definition 3.4) as well as all threshold graphs (see Definition 3.9) satisfy Conjecture 1.1.*

We also show that wedges of such graphs satisfy Conjecture 1.1 (see Corollary 3.10).

2 Preliminaries

A graph G is an ordered pair of sets $(V(G), E(G))$ where $V(G)$ is called the set of *vertices* and $E(G)$ is the set of *edges*. Each element of $E(G)$ is a subset of $V(G)$ of size 2. We say that two vertices v, w are *adjacent* if $\{v, w\} \in E(G)$. For any vertex $v \in V(G)$, we define its *neighborhood* to be the set $N(v) := \{w \in V(G) \mid \{v, w\} \in E(G)\}$. The number of vertices adjacent to v (i.e., $\#N(v)$) is called the *degree* of v .

For any positive integer n , we use $[n]$ to denote the set $\{1, 2, \dots, n\}$. Set P_n to be the path graph on n vertices, i.e., $V(P_n) = [n]$ and $E(P_n) = \{\{k, k+1\} \mid k \in [n-1]\}$. The *cycle graph* on n vertices C_n is the graph obtained from P_n by adding the edge $\{1, n\}$.

A *subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We sometimes simply say that G *contains* H . For any set of vertices $W \subseteq V(G)$, we denote by $G \setminus W$ the subgraph of G with vertex set $V(G) \setminus W$ and edge set $\{\{v, w\} \in E(G) \mid v, w \notin W\}$. We say that we have *deleted* the vertices in W . A graph is said to be *connected* if for any two vertices v, w , the graph contains a path whose endpoints are v and w . The *connected components* of a graph are its maximal connected subgraphs.

We now recall a few results from [5]. First, we describe the forcing sets in path graphs.

Result 2.1. [5, Proposition 5] *For any $n \geq 1$, we have $S \subseteq [n]$ is non-forcing in P_n if and only if $1, n \notin S$ and S does not contain consecutive numbers. Hence, for any $i \geq 1$,*

$$z'(P_n; i) = \binom{n-i-1}{i} \text{ and } z(P_n; i) = \binom{n}{i} - \binom{n-i-1}{i}.$$

Before moving ahead, we recall the following notion, which helps in identifying certain graphs satisfying Conjecture 1.1.

Definition 2.2. *A fort in a graph G is a non-empty set of vertices F such that no vertex outside F is adjacent to exactly one vertex in F . That is, for any vertex $v \notin F$, we have $\#(N(v) \cap F) \neq 1$.*

Note that if $S \subset V(G)$ is such that $S \cap F = \emptyset$ for a fort F , then S is a non-forcing set of G . This is because no vertex in F can be colored. Using this fact, one can prove the following result.

Result 2.3. [5, Proposition 17] *If a graph G has a fort F such that $\#F \leq z(G) + 1$, then G satisfies Conjecture 1.1.*

We also have the following result for the *union* of two graphs, which can be derived from [5, Proposition 13]. For two graphs G and H (with disjoint vertex sets), we define their union $G \sqcup H$ to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Result 2.4. *If G and H satisfy Conjecture 1.1, then so does $G \sqcup H$.*

Proof. It can be checked that it is enough to prove the result for when G and H are both path graphs. Let $m, n \geq 1$. Consider the path graph P_{m+n} whose vertex set is $[m+n]$ and consider the subgraph obtained by deleting the edge $\{m, m+1\}$. This subgraph is clearly the same as $P_m \sqcup P_n$. It is easy to see that any forcing set in $P_m \sqcup P_n$ is also forcing in P_{m+n} and hence $z(P_m \sqcup P_n; i) \leq z(P_{m+n}; i)$ for all $i \geq 1$. $\square$

3 Bounds via graph operations

In this section, we consider various operations on graphs such as deleting or adding vertices and edges, and examine how they affect the number of zero forcing sets. We then use these results to show that various classes of graphs satisfy Conjecture 1.1.

3.1 Leaves and hanging cycles

A *leaf* in a graph is a vertex of degree 1. We have the following result for graphs that have a leaf.

Lemma 3.1. *Let G be a graph and x be a leaf that is adjacent to a vertex v . For any $i \geq 1$, we have*

$$z'(G; i) \geq z'(G \setminus \{x\}; i) + z'(G \setminus \{x, v\}; i - 1).$$

In particular, if $G \setminus \{x\}$ and $G \setminus \{x, v\}$ both satisfy Conjecture 1.1, then so does G .

Proof. We prove the result by describing an injection

$$\iota : Z'(G \setminus \{x\}; i) \cup Z'(G \setminus \{x, v\}; i - 1) \hookrightarrow Z'(G, i).$$

Let $S \in Z'(G \setminus \{x\}; i)$. If $v \in S$, we set $\iota(S) = S \setminus \{v\} \cup \{x\}$. The fact that $\iota(S) \in Z'(G; i)$ follows because the only vertex that x can force is v . If $v \notin S$, we set $\iota(S) = S$, which is non-forcing in G as well.

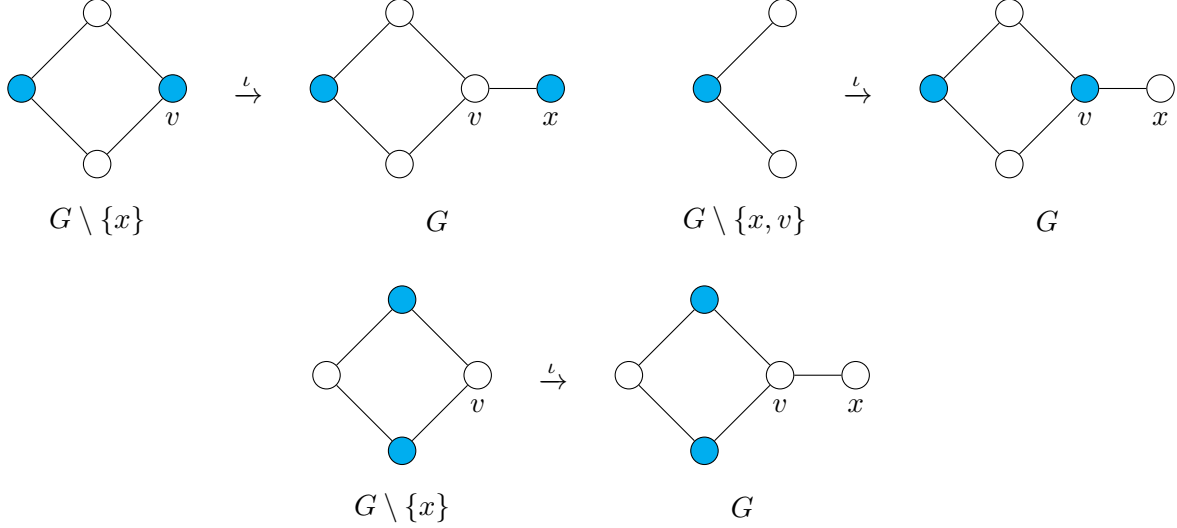


Figure 3: Instances of the injection ι .

Let $S \in Z'(G \setminus \{x, v\}; i - 1)$. We set $\iota(S) = S \cup \{v\}$. Since $x \notin \iota(S)$, the vertex v can, at most, be used to color x . But in this case, all neighbors of v (apart from x) have to already be colored. Hence, the vertex v cannot be used to color more vertices in $G \setminus \{x, v\}$ than S can. This shows that $\iota(S) \in Z'(G; i)$.

It can be verified that ι is an injection. Now suppose that G has n vertices and $G \setminus \{x\}$ and $G \setminus \{x, v\}$ satisfy Conjecture 1.1. We have for any $i \geq 1$,

$$\begin{aligned} z'(G; i) &\geq z'(G \setminus \{x\}; i) + z'(G \setminus \{x, v\}; i - 1) \\ &\geq z'(P_{n-1}; i) + z'(P_{n-2}; i - 1) \\ &= \binom{(n-1) - i - 1}{i} + \binom{(n-2) - (i-1) - 1}{i-1} = \binom{n-i-1}{i} = z'(P_n; i). \end{aligned}$$

Hence, G satisfies Conjecture 1.1. $\square$

Recall that a *tree* is a connected graph that contains no cycles. Since any tree has a leaf, induction on the number of vertices in a tree (along with Result 2.4) gives us the following result.

Corollary 3.2. *For any tree T on n vertices, $z(T; i) \leq z(P_n; i)$ for all $i \geq 1$.*

The above corollary is a strengthening of [13, Theorem 1.6] and also proves a conjecture in Sam Spiro's list of open problems [26, Conjecture 3.7].

We now consider deletion of an edge in certain special cycles in a graph. We say that a graph has a *hanging cycle* on an edge $e = \{v, w\}$ if it has a cycle $v, a_1, \dots, a_k, w, v$ such that for all $i \in [k]$, the vertex a_i has degree 2. There is no constraint on the degree of the vertices v and w .

Lemma 3.3. *Let G be a graph with a hanging cycle on an edge e . Let G' be the graph obtained from G by removing the edge e . We have $z(G; i) \leq z(G'; i)$ for all $i \geq 1$. In particular, if G' satisfies Conjecture 1.1, so does G .*

Proof. We will in fact show that $Z(G; i) \subseteq Z(G'; i)$ for all $i \geq 1$. Let S be a forcing set in G . We have to show that S is forcing in G' as well.

Suppose $e = \{v, w\}$ and $v, a_1, a_2, \dots, a_k, w$ is the cycle hanging from the edge e . If S can color G without using the edge e to force a vertex, then it is a forcing set in G' as well. Suppose we use the edge e in the coloring. Note that the edge e only plays a role in the forcing if one of v or w is used to force the other.

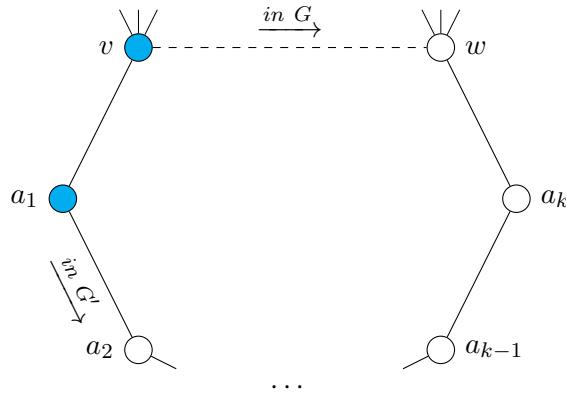


Figure 4: Forcing sets remain forcing after deleting e .

Without loss of generality, suppose v and all its neighbors in G apart from w are colored at some point in the forcing. Then we can use e to color w in G . However, in G' , we can use a_1 (which is colored since it is a neighbor of v) to color a_2 , which colors a_3 , and so on. Finally, coloring along the cycle, we can color w . Hence, S is a forcing set in G' as well. $\square$

The lemmas above allow us to prove that certain planar graphs satisfy Conjecture 1.1. Recall that a planar graph is one that can be drawn on the plane with no edges crossing.

Definition 3.4. An outerplanar graph is a planar graph which can be drawn on the plane in such a way that all the vertices lie on the outer (unbounded) face (see [27] for more details).

For example, the graph in Figure 5 is outerplanar whereas the complete graph K_4 , although planar, is not outerplanar. In particular, all trees are outerplanar.

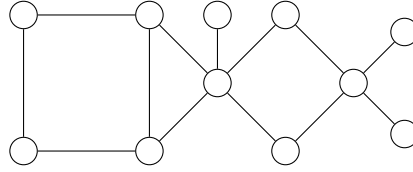


Figure 5: An outerplanar graph.

To show that all outerplanar graphs satisfy Conjecture 1.1, we use the fact that any outerplanar graph has either a leaf or a hanging cycle. To prove this, we first need a few definitions.

A *cut vertex* of a connected graph is a vertex whose deletion results in a disconnected graph. A graph is said to be *biconnected* (or 2-connected) if it is connected and has no cut vertices. The maximal biconnected subgraphs of a connected graph are called its *blocks*. Cut vertices are precisely those vertices that lie in more than one block.

Biconnected outerplanar graphs are relatively simple to describe. Such graphs with at most two vertices are trivial to characterize. Biconnected outerplanar graphs with at least three vertices are precisely those graphs obtained from cycle graphs by adding any number of non-crossing edges between the vertices (see [27]).

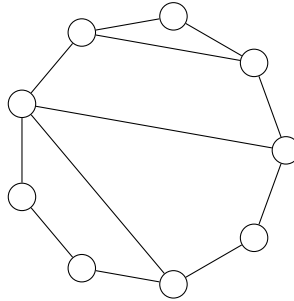


Figure 6: A biconnected outerplanar graph.

We will need another definition before proving that outerplanar graphs satisfy Conjecture 1.1. The *block-cut tree* of a connected graph G is a tree whose vertices correspond to

the blocks and cut vertices of G . There is an edge between a block and a cut vertex if the cut vertex lies in the block. It can be verified that the graph defined does indeed form a tree.

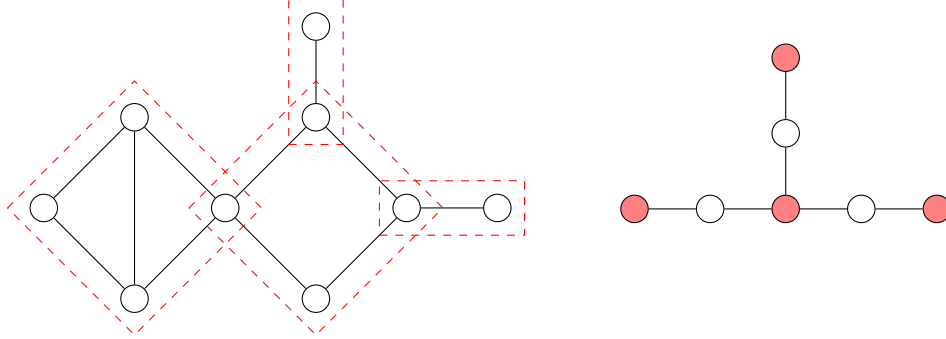


Figure 7: A graph and its block-cut tree with blocks specified in red.

Theorem 3.5. *All outerplanar graphs satisfy Conjecture 1.1.*

Proof. Let O be an outerplanar graph. We use induction on the sum of the number of vertices and edges of O . By Result 2.4, we can assume that O is connected. If O has a leaf, we use Lemma 3.1 and the induction hypothesis. Note that any subgraph of an outerplanar graph is outerplanar.

Next suppose that O has no leaves. If O is biconnected, then it is obtained from a cycle graph by adding non-crossing edges between the vertices. Hence, it is clear that O will have a hanging cycle and we use Lemma 3.3 and the induction hypothesis.

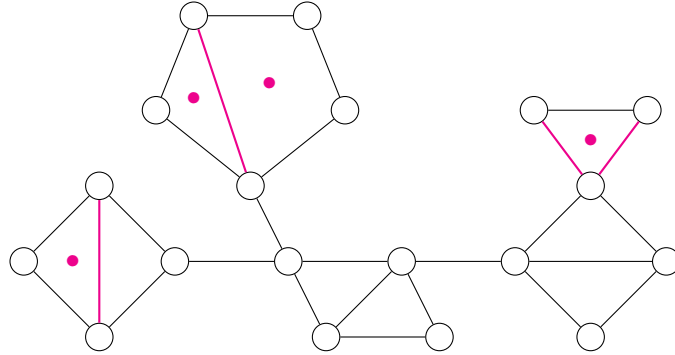


Figure 8: Outerplanar graph with hanging cycles in ‘leaf blocks’ specified (along with edge(s) they hang on). Note that there are other hanging cycles as well.

If O is not biconnected, consider a block B of O that has only one cut vertex v of O (this is a block that corresponds to a leaf in the block-cut tree of O). Note that the subgraph B is a biconnected outerplanar graph and hence is obtained from a cycle graph by adding non-crossing edges between the vertices. We can always find a hanging cycle in the subgraph B that is either hanging by an edge incident to v or one that does not contain the vertex v . Such a hanging cycle is a hanging cycle of O as well and we use Lemma 3.3 and the induction hypothesis. $\square$

The idea used in the proof of the above theorem in fact gives us a slightly more general result. Let G and H be graphs with disjoint vertex sets. Let $v \in V(G)$ and $w \in V(H)$. The wedge of H with G at v (using w) is the graph obtained from $G \sqcup H$ by identifying the vertices v and w . When the vertex $v \in V(G)$ or $w \in V(H)$ used in the identification does not matter, it is not mentioned.

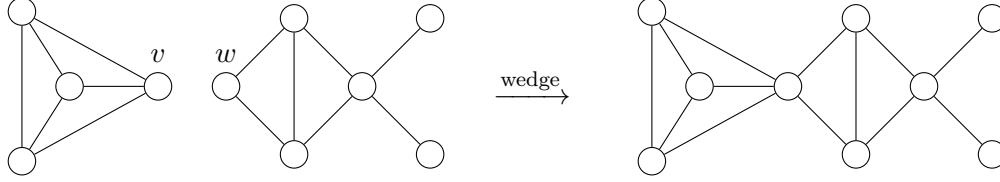


Figure 9: Wedge of two graphs.

Corollary 3.6. *Let G be a graph with a vertex v such that both G and $G \setminus \{v\}$ satisfy Conjecture 1.1. Any graph obtained from G by wedging an outerplanar graph at v also satisfies Conjecture 1.1.*

Proof. Let O be the outerplanar graph we wedge at v . We use induction on the sum of the number of vertices and edges of O . If O has a leaf (other than v), then we use Lemma 3.1 and apply the induction hypothesis. Note that even if such a vertex is adjacent to v , deleting v gives a disjoint union of $G \setminus \{v\}$ and an outerplanar graph and Result 2.4 shows that this union also satisfies Conjecture 1.1. Next, suppose that O has no such leaf. Using similar ideas as in the proof of Theorem 3.5, it can be shown that O must have a hanging cycle that is also a hanging cycle in the wedge. In this case, we apply Lemma 3.3. $\square$

For example, the above corollary can be used to show that the wedged graph in Figure 9 satisfies Conjecture 1.1.

The following is an immediate corollary to Lemma 3.3.

Corollary 3.7. *Let G be a graph and $v \in V(G)$. Let G' be the graph obtained by wedging the cycle graph C_n with G at v . Let G'' be the graph obtained by wedging the path P_n with G at v using a leaf vertex of the path. Then $z(G'; i) \leq z(G''; i)$ for all $i \geq 1$.*

In some sense, the graph G'' in the above corollary is more ‘path-like’ than G' . We will see another result (Proposition 3.13) where a more ‘path-like’ graph has more forcing sets of each cardinality than one that is less ‘path-like’.

3.2 Cones and simplicial vertices

For any graph G , the *cone* over G , denoted by $c(G)$, is the graph obtained by adding a new vertex v that is adjacent to all vertices of G . Here v is called the cone vertex.

Lemma 3.8. *Let G be a graph and $c(G)$ be the cone over G with cone vertex v .*

1. *If G has no isolated vertices and satisfies Conjecture 1.1, then so does $c(G)$.*
2. *If G has exactly one isolated vertex x and $G \setminus \{x\}$ satisfies Conjecture 1.1, then so does $c(G)$.*

3. If G has at least 2 isolated vertices, then $c(G)$ satisfies Conjecture 1.1.

Proof. 1. To prove the first statement, we note that if $S \in Z'(G; i)$ for some $i \geq 1$, then $S \in Z'(c(G); i)$ and $S \cup \{v\} \in Z'(c(G); i + 1)$. This is because, assuming v has been colored at some point, it can only force a vertex w in G if all other vertices in G are already colored. But this would mean that $S \in Z(G; i)$ since w could be forced by any of its neighbors in G .

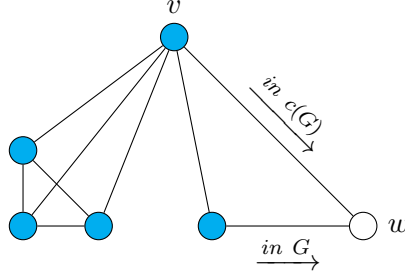


Figure 10: Vertex forced by v in $c(G)$ can be forced in G as well.

Let n be the number of vertices in G . Since G satisfies Conjecture 1.1, this shows that for any $i \geq 1$,

$$\begin{aligned}
 z'(c(G); i) &\geq z'(G; i) + z'(G; i - 1) \\
 &\geq z'(P_n; i) + z'(P_n; i - 1) \\
 &= \binom{n - i - 1}{i} + \binom{n - (i - 1) - 1}{i - 1} \\
 &\geq \binom{n - i - 1}{i} + \binom{n - i - 1}{i - 1} = \binom{(n + 1) - i - 1}{i} = z'(P_{n+1}; i).
 \end{aligned}$$

Hence, $c(G)$ also satisfies Conjecture 1.1.

2. The second statement follows from Lemma 3.1 and the first statement. Note that x becomes a leaf in $c(G)$.
3. The third statement follows from Result 2.3 since any two isolated vertices in G form a fort in $c(G)$.

□

In particular, the above lemma shows, by induction on the number of vertices, that all *threshold graphs* satisfy Conjecture 1.1.

Definition 3.9. Threshold graphs are defined recursively as follows:

1. The graph consisting of just one vertex is a threshold graph.
2. If G is a threshold graph, so is $c(G)$.
3. If G is a threshold graph, so is the graph obtained by adding an isolated vertex to G .

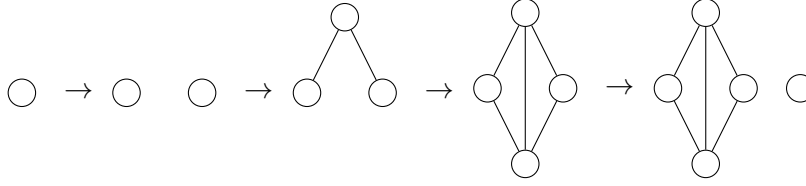


Figure 11: Construction of a threshold graph on 5 vertices.

We note that the zero forcing polynomial of threshold graphs has been studied in [5, Section 4.2].

Corollary 3.10. *Let G be a threshold graph and $v_1, v_2, \dots, v_k$ be vertices of G . Let $O_1, O_2, \dots, O_k$ be outerplanar graphs. The graph obtained by wedging the graph O_i with G at v_i for all $i \in [k]$ satisfies Conjecture 1.1.*

Proof. Note that deleting a vertex from a threshold (respectively, outerplanar) graph results in a threshold (respectively, outerplanar) graph. The result now follows using induction on k and Corollary 3.6. $\square$

Similar to hanging cycles, we now consider another operation where we delete certain edges of a graph. A vertex v in a graph is called a *simplicial vertex* if all the vertices adjacent to v are adjacent to each other, i.e., its neighborhood $N(v)$ is a clique.

Lemma 3.11. *Let G be a graph with a simplicial vertex v . Let G' be a graph obtained from G by deleting some of the edges between vertices of $N(v)$. For all $i \geq 1$, we have $z(G; i) \leq z(G'; i)$.*

Proof. We show that $Z(G; i) \subseteq Z(G'; i)$ for all $i \geq 1$. Let S be a forcing set in G . Suppose the edge $e = \{w_1, w_2\}$ where $w_1, w_2 \in N(v)$ plays a role in the coloring of G . This happens only if, without loss of generality, at some point in the coloring, w_1 and all its neighbors apart from w_2 are colored and w_1 is used to color w_2 . Note that in this case, none of the other edges between vertices of $N(v)$ played a role in the coloring up to that point since w_2 was uncolored.

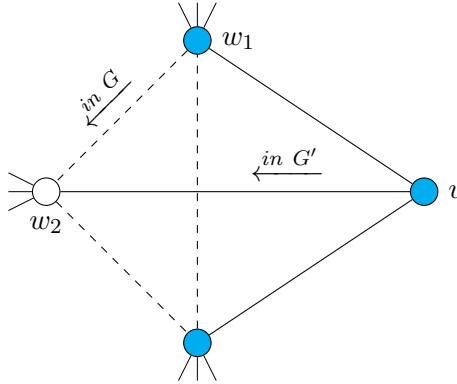


Figure 12: Forcing sets in G are forcing in G' .

Instead of using w_1 to color w_2 , we note that v is colored, and all the vertices in $N(v)$ apart from w_2 are colored. Hence, in G' , v can be used to color w_2 . This shows that S is forcing in G' as well. $\square$

Corollary 3.12. *Let G be a graph that satisfies Conjecture 1.1 and $v \in V(G)$. The graph obtained by adding all edges between the vertices of $N(v)$ (thus making v a simplicial vertex) also satisfies Conjecture 1.1.*

3.3 Hanging paths

A *hanging path* at a vertex v of a graph G is a path $a_1, a_2, \dots, a_k$ in G such that

- a_1 is adjacent to v ,
- for all $i \in [k - 1]$, the vertex a_i has degree 2, and
- the vertex a_k has degree 1.

We now show that ‘combining’ the paths hanging from a vertex increases the number of forcing sets of each cardinality. One way to interpret this result is that graphs that are more ‘path-like’ have more zero forcing sets of each cardinality.

Proposition 3.13. *Suppose that a graph G has two paths A and B hanging at a vertex v . Let G' be the graph obtained from G by removing B and attaching it at the end of A (see Figure 13). For all $i \geq 1$, we have $z(G; i) \leq z(G'; i)$.*

Proof. Suppose that the path A has vertices $a_1, a_2, \dots, a_k$ and the path B has vertices $b_1, b_2, \dots, b_m$ with the vertices a_1, b_1 adjacent to v . The graph G' is obtained from G by deleting the edge $\{v, b_1\}$ and adding the edge $\{a_k, b_1\}$. We use C to denote the path $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m$ hanging from v in G' .

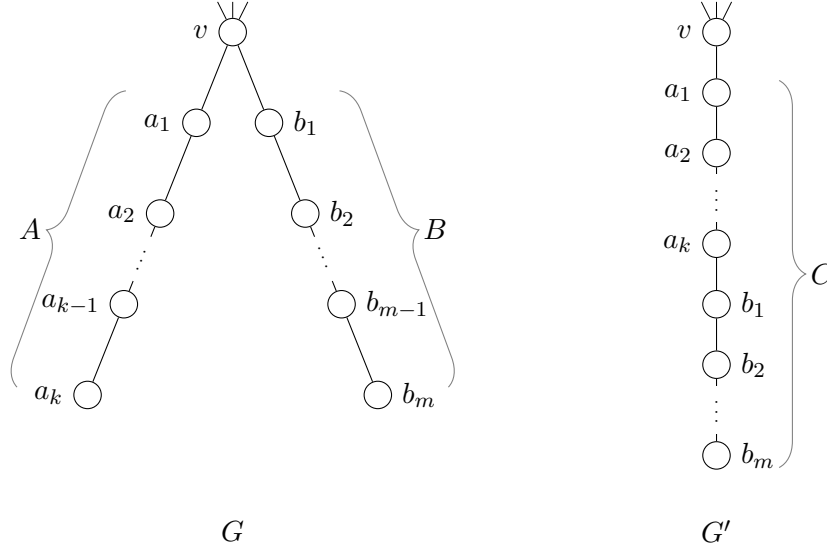


Figure 13: The graphs G and G' of Proposition 3.13.

Fix $i \geq 1$. We prove the result by describing an injection $\iota : Z(G; i) \hookrightarrow Z(G'; i)$. Suppose $S \in Z(G; i)$.

Case 1: We first assume that $v \in S$. In this case, we set $\iota(S) = S$ (note that $V(G) = V(G')$). We have to show that $S \in Z(G'; i)$.

First, we consider the case when, in G , all neighbors of v not in A or B can be colored without using v to force any coloring. Then the same is true in G' as well. However, in G' , this implies that all vertices in C can now be colored since all neighbors of v apart from a_1 are colored. Hence, v forces a_1 , which in turn forces a_2 , and so on. This shows that $S \in Z(G'; i)$.

Next, suppose that in G , the vertex v has to be used to color one of its neighbors that is not in A or B . This means that the vertices a_1 and b_1 can be colored in G without using the vertex v . This, in turn, means that a_1 was colored using the vertices in $S \cap A$ and b_1 was colored using the vertices in $S \cap B$. To show that $S \in Z(G'; i)$, we just have to show that $S \cap C$ can force the vertex a_1 in the path graph C . If $a_1 \in S$, this is trivially true. If $a_1 \notin S$, then either a consecutive pair of vertices of A or the vertex a_k belongs to S (see Result 2.1). If a consecutive pair of vertices of A is in S , then $S \cap C$ forces a_1 . Assuming that there is no such pair, we have $a_k \in S$. Now, since b_1 can be colored in B using $S \cap B$, the same method can be used to color b_1 in $S \cap C$. Hence, both b_1 and a_k can be colored, and we get a consecutive pair of colored vertices in C , which can be used to force a_1 .

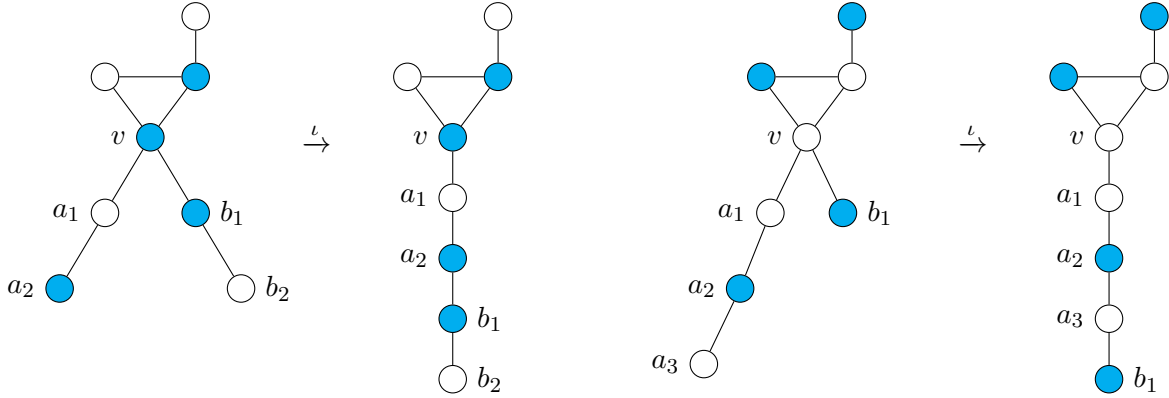


Figure 14: Instances of ι corresponding to **Case 1** and **Case 2**.

Case 2: Suppose that $v \notin S$ and v can be forced in G using neither a_1 nor b_1 . In this case as well, we set $\iota(S) = S$ and hence we have to show that $S \in Z(G'; i)$.

This case is very similar to the previous case. Note that any coloring done in $G \setminus (A \cup B)$ can be done in $G' \setminus C$ as well. Our assumption says that S can be used to color v in $G \setminus (A \cup B)$ and hence in $G' \setminus C$. Now we can assume v is colored and this brings us back to the previous case.

Case 3: Suppose that $v \notin S$ and v can only be forced in G using a_1 or b_1 . This means that S restricted to the path in G induced by the vertices $A \cup \{v\}$ or the path induced by $B \cup \{v\}$ is a forcing set (where v is not colored). Hence, one of the following must hold:

- There are two consecutive vertices in $S \cap A$ or $a_k \in S$.
- There are two consecutive vertices in $S \cap B$ or $b_m \in S$.

In all cases *apart* from when

- there are no consecutive vertices in $S \cap A$ or $S \cap B$, and
- $a_k \in S$ and $b_1, b_m \notin S$,

we define $\iota(S) = S$. It can be checked that in all cases apart from the one mentioned above, S restricted to the path $C \cup \{v\}$ is a forcing set. This gives us that $S \in Z(G'; i)$.

In the case mentioned above, we define $\iota(S)$ by ‘reversing the coloring on $A \cup \{v\}$ ’. We identify any subset X of $V(G)$ (or $V(G')$) with the choice function

$$c_X(w) = \begin{cases} 1, & \text{if } w \in X \\ 0, & \text{if } w \notin X. \end{cases}$$

Now, setting $a_0 := v$, we define $\iota(S)$ by

- $c_{\iota(S)}(a_l) = c_S(a_{k-l})$ for all $0 \leq l \leq k$, and
- $c_{\iota(S)}(w) = c_S(w)$ for all $w \in V(G \setminus (A \cup \{v\}))$.

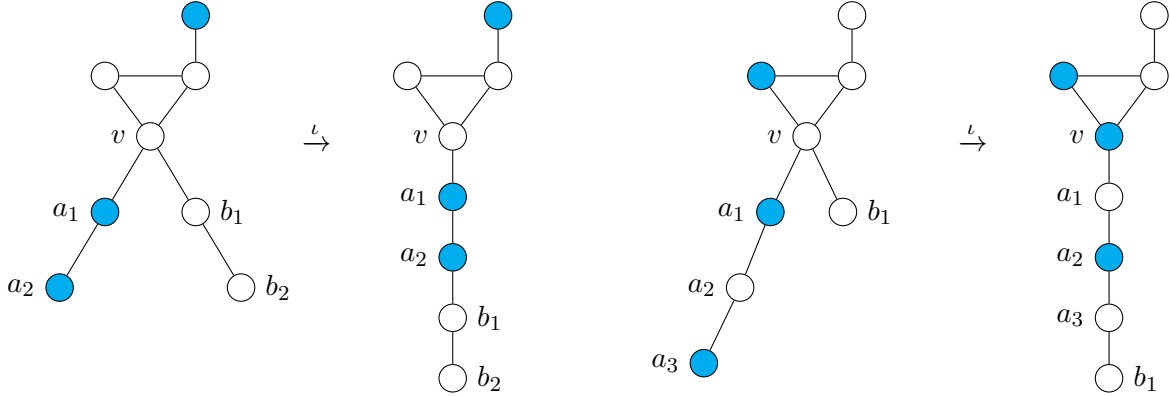


Figure 15: Instances of the injection ι corresponding to **Case 3**.

We have to show that $\iota(S) \in Z(G'; i)$. Note that for S in the graph G , the vertex v has an uncolored neighbor b_1 . Also, by the conditions on S , b_1 can only be colored using v . This means that, at least after v has been colored, all the neighbors of v apart from b_1 can be colored without using the vertex v to force any coloring.

Looking at $\iota(S)$ in G' , note that since $v \in \iota(S)$ and $\iota(S) \setminus C = S \setminus (A \cup B)$, all the neighbors of v in G' apart from a_1 can be colored without using v to force any coloring. This gives us that v can be used to force a_1 , which forces a_2 , and so on. Hence, $\iota(S) \in Z(G'; i)$. It can also be checked that in this case, $\iota(S) \notin Z(G; i)$ (neither a_1 nor b_1 can be colored). Hence, ι is an injection, as required. $\square$

Let T be a tree with n vertices. A pleasing consequence of the above proposition is that repeated application of its proof gives an injection $Z(T; i) \hookrightarrow Z(P_n; i)$ for all $i \geq 1$. This gives an alternate proof of Corollary 3.2. In fact, we can prove the slightly stronger result given below, which was suggested by Sam Spiro [25].

Corollary 3.14. *Let T be a tree on n vertices that is not the path graph. For any $1 \leq i < \frac{n}{2}$, we have $z(T; i) < z(P_n; i)$.*

Proof. It can be checked that path graphs are the only graphs that have forcing sets of size 1. Hence, the result is true for $i = 1$ and we have to prove it for $2 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$. Repeatedly applying Proposition 3.13, we see that it is enough to prove the result for the case when T has a vertex v with two hanging paths A and B such that $T \setminus (A \cup B)$ is a path graph.

To prove the result, we show that the injection $\iota : Z(T; i) \hookrightarrow Z(P_n; i)$ (defined in the proof of Proposition 3.13) is not surjective. We label the vertices of A and B just as before and label the vertices in the path graph $T \setminus (A \cup B \cup \{v\})$ as $c_1, c_2, \dots, c_p$ where c_p is adjacent to v . Hence, the path graph P_n we consider is the one with vertices labeled $c_1, c_2, \dots, c_p, v, a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m$.

Consider any $S \subset V(P_n)$ that satisfies the following conditions:

- Both $v, a_1 \in S$.
- When S is restricted to the path induced by $\{b_1, b_2, \dots, b_m\}$, we have a non-forcing set. In particular, $b_1 \notin S$.
- When S is restricted to the path induced by $\{c_1, c_2, \dots, c_p\}$, we have a non-forcing set. In particular, $c_p \notin S$.

Since $v, a_1 \in S$, it is clear that any such S is forcing in P_n . Note that if we consider S (or S with the coloring on $A \cup \{v\}$ ‘reversed’) as a subset of T , then neither b_1 nor c_p can be colored and hence S is non-forcing in T .

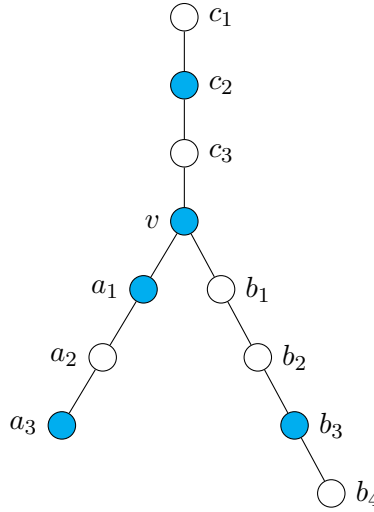


Figure 16: A non-forcing set in T that is forcing in P_n .

Since the largest size of any non-forcing set in a path graph with l vertices is $\lfloor \frac{l-1}{2} \rfloor$, we can construct such an S of size i for all i such that $2 \leq i \leq \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{p-1}{2} \rfloor + k + 1$. Using

the fact that $k, m, p \geq 1$ and that for any $r, s \geq 1$, $\lfloor \frac{r-1}{2} \rfloor + \lfloor \frac{s-1}{2} \rfloor + 1 \geq \lfloor \frac{r+s-1}{2} \rfloor$, we have

$$\begin{aligned} \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor + k + 1 &\geq \left\lfloor \frac{m+p-1}{2} \right\rfloor + k \\ &\geq \left\lfloor \frac{m+p-1}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor + 1 \geq \left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned}$$

This proves the required result. $\square$

4 Future directions

Upon having Corollary 3.2, one would hope that Conjecture 1.1 can be proved by showing that any connected graph G has a spanning tree T such that $z(G; i) \leq z(T; i)$ for all $i \geq 1$. Unfortunately, this is not true. It can be checked that the graph in Figure 17 does not have a spanning tree with this property.

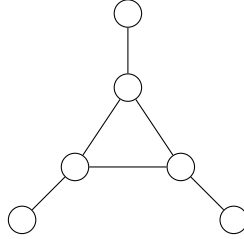


Figure 17: A graph G without a spanning tree T such that $[G] \leq [T]$.

However, this does lead to asking whether, given a graph G , one can find a ‘nice’ graph H such that $z(G; i) \leq z(H; i)$ for all $i \geq 1$. Ideally, H should be a graph that satisfies Conjecture 1.1 and has the same number of vertices as G .

One way to address this question, as we have done in this article, is to consider graph operations that increase the number of forcing sets of each cardinality. For example, some other operations that could be studied are: subdivision of an edge, product of graphs, and wedge of graphs.

Another direction is to consider the following equivalence and poset structure on graphs: We say that two graphs G and H with n vertices are equivalent if $z(G; i) = z(H; i)$ for all $i \geq 1$. Denote the equivalence class of a graph G by $[G]$. This equivalence has been studied in [5, Section 5.1]. For example, they show that $[P_n] = \{P_n\}$. We define a poset structure on these equivalence classes by setting $[G] \leq [H]$ if $z(G; i) \leq z(H; i)$ for all $i \geq 1$. Note that Conjecture 1.1 asserts that $[P_n]$ is the maximum element of this poset.

What else can be said about this poset? For example, what are the graphs G such that $[G] \leq [C_n]$? If $[P_n]$ is indeed the maximum of this poset, what can be said about the coatoms, i.e., the elements covered by $[P_n]$?

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References

- [1] J. S. Alameda. Leaky forcing: a new variation of zero forcing. *Australas. J. Combin.*, 88:308–326, 2024.
- [2] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra Appl.*, 433(2):401–411, 2010.
- [3] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Parameters related to tree-width, zero forcing, and maximum nullity of a graph. *J. Graph Theory*, 72(2):146–177, 2013.
- [4] K. F. Benson, D. Ferrero, M. Flagg, V. Furst, L. Hogben, V. Vasilevska, and B. Wissman. Zero forcing and power domination for graph products. *Australas. J. Combin.*, 70:221–235, 2018.
- [5] K. Boyer, B. Brimkov, S. English, D. Ferrero, A. Keller, R. Kirsch, M. Phillips, and C. Reinhart. The zero forcing polynomial of a graph. *Discrete Appl. Math.*, 258:35–48, 2019.
- [6] C. Bozeman, B. Brimkov, C. Erickson, D. Ferrero, M. Flagg, and L. Hogben. Restricted power domination and zero forcing problems. *J. Comb. Optim.*, 37(3):935–956, 2019.
- [7] B. Brimkov and I. V. Hicks. Complexity and computation of connected zero forcing. *Discrete Appl. Math.*, 229:31–45, 2017.
- [8] D. Burgarth, D. D’Alessandro, L. Hogben, S. Severini, and M. Young. Zero forcing, linear and quantum controllability for systems evolving on networks. *IEEE Trans. Automat. Control*, 58(9):2349–2354, 2013.
- [9] D. Burgarth and V. Giovannetti. Full control by locally induced relaxation. *Phys. Rev. Lett.*, 99(10):100501, 2007.
- [10] S. Butler, J. Grout, and H. T. Hall. Using variants of zero forcing to bound the inertia set of a graph. *Electron. J. Linear Algebra*, 30:1–18, 2015.
- [11] B. Cameron, J. Janssen, R. Matthew, and Z. Zhang. An approximation algorithm for zero forcing. *arXiv preprint arXiv:2402.08866*, 2024.
- [12] Y. Caro and R. Pepper. Dynamic approach to k -forcing. *Theory Appl. Graphs*, 2(2):Art. 2, 5, 2015.
- [13] B. Curtis, L. Gan, J. Haddock, R. Lawrence, and S. Spiro. Zero forcing with random sets. *Discrete Math.*, 347(5):Paper No. 113944, 2024.

- [14] P. A. Dreyer, Jr. and F. S. Roberts. Irreversible k -threshold processes: graph-theoretical threshold models of the spread of disease and of opinion. *Discrete Appl. Math.*, 157(7):1615–1627, 2009.
- [15] J. Ekstrand, C. Erickson, H. T. Hall, D. Hay, L. Hogben, R. Johnson, N. Kingsley, S. Osborne, T. Peters, J. Roat, A. Ross, D. D. Row, N. Warnberg, and M. Young. Positive semidefinite zero forcing. *Linear Algebra Appl.*, 439(7):1862–1874, 2013.
- [16] S. English, C. MacRury, and P. Prał at. Probabilistic zero forcing on random graphs. *European J. Combin.*, 91:Paper No. 103207, 22, 2021.
- [17] J. Geneson, R. Haas, and L. Hogben. Reconfiguration graphs of zero forcing sets. *Discrete Appl. Math.*, 329:126–139, 2023.
- [18] F. Goldberg and A. Berman. Zero forcing for sign patterns. *Linear Algebra Appl.*, 447:56–67, 2014.
- [19] A. M. R.-S. G. W. Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra Appl.*, 428(7):1628–1648, 2008.
- [20] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, and M. A. Henning. Domination in graphs applied to electric power networks. *SIAM J. Discrete Math.*, 15(4):519–529, 2002.
- [21] L. Hogben, K. F. Palmowski, D. E. Roberson, and M. Young. Fractional zero forcing via three-color forcing games. *Discrete Appl. Math.*, 213:114–129, 2016.
- [22] L.-H. Huang, G. J. Chang, and H.-G. Yeh. On minimum rank and zero forcing sets of a graph. *Linear Algebra Appl.*, 432(11):2961–2973, 2010.
- [23] C. X. Kang and E. Yi. Probabilistic zero forcing in graphs. *Bull. Inst. Combin. Appl.*, 67:9–16, 2013.
- [24] D. D. Row. Zero forcing number, path cover number, and maximum nullity of cacti. *Involve*, 4(3):277–291, 2011.
- [25] S. Spiro. Personal communication, 2024.
- [26] S. Spiro. Problems that I would like Somebody to Solve, Version of August 7, 2023. <https://samspiro.xyz/Research/P2Solve.pdf>.
- [27] M. M. Sysłó. Characterizations of outerplanar graphs. *Discrete Math.*, 26(1):47–53, 1979.
- [28] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.5.0)*, 2022. <https://www.sagemath.org>.
- [29] B. Yang. Fast-mixed searching and related problems on graphs. *Theoret. Comput. Sci.*, 507:100–113, 2013.